Completeness of $\mathbb{R}$

① Least upper bound property.

> Every nonempty subset of $\mathbb{R}$ having an upper bound must have a supremum.

② Nested intervals Property.

> If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$, is a nested sequence of closed bounded interals, then $\exists\, \xi \in \mathbb{R}$ s.t. $\xi \in I_n \; \forall n \in \mathbb{N}$.

Indeed, ① is equivalent with ②.

① $\Rightarrow$ ② (Proved in Lecture)

Eg. $I_n = [0, \frac{1}{n}]$. Then $\xi = 0 \in I_n \; \forall n \in \mathbb{N}$.

② ⇒ ①: Assume NIP. Need to prove LUBP.

Let X be a nonempty subset of $\mathbb{R}$ and

X be bounded above by K.

Since X is nonempty, we can find some $x \in X$.

Without loss of generality, we may assume $x = 0 \in X$.

Let $I_0 := [0, K]$.

If $\frac{K}{2}$ is an upper bound for I_1, then define $I_2 := [0, \frac{K}{2}]$.

Otherwise, define $I_1 := [\frac{K}{2}, K]$.

Continuing this process, we can find a nested sequence

of intervals. $I_n := [a_n, b_n]$ s.t. $\forall n \in \mathbb{N}$,

① b_n is an upper bound for X.

② $\exists x_n \in X$ s.t. $x_n \in [a_n, b_n]$. (Check this!)

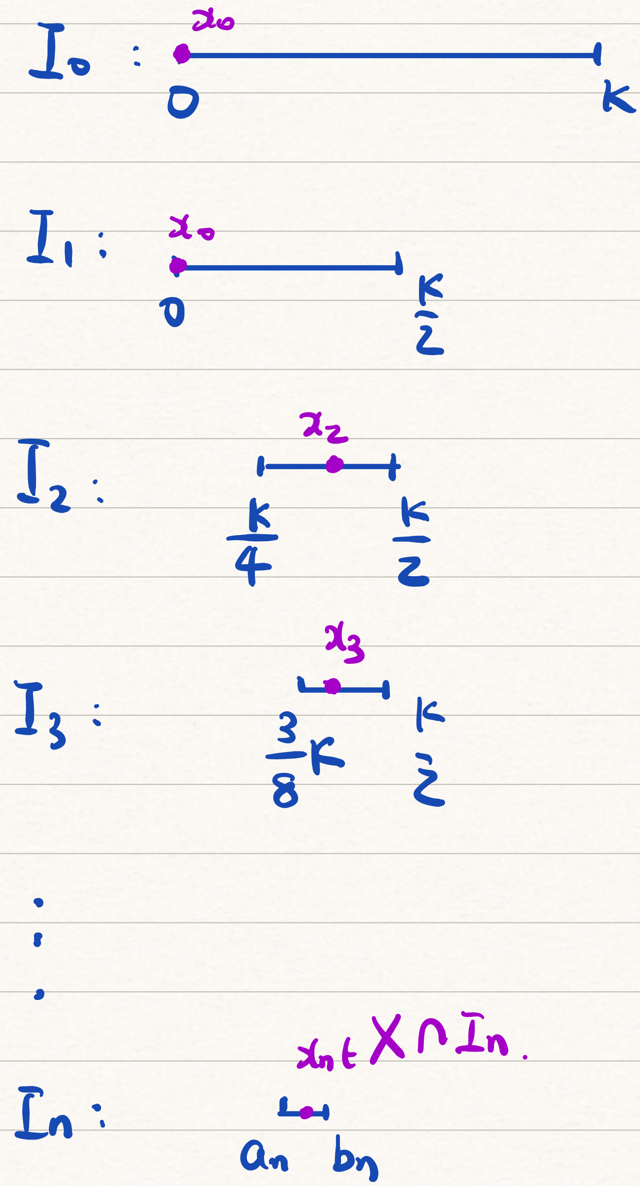
③ Length of $I_n = (b_n - a_n) = \frac{K}{2^n}$.

By NIP (②), $\exists \xi \in \mathbb{R}$ s.t. $\xi \in I_n \; \forall n \in \mathbb{N}$.

And since $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$, (Theorem 2.5.3)

then ξ contained in $I_n \; \forall n$ is unique.

Exercise: Prove ξ is the supremum of X.


I_0 : x_0 0 K
I_1 : x_0 0 $\frac{K}{2}$
I_2 : x_2 $\frac{K}{4}$ $\frac{K}{2}$
I_3 : x_3 $\frac{3}{8}K$ $\frac{K}{2}$
$\vdots$
$x_n \in X \cap I_n$.
I_n : a_n b_n

- Application of the supremum Property

Exercise: (Existence of $\sqrt[n]{a}$).

Let $a>0$. Show that $\forall n\in\mathbb{N}$, there exists a unique positive number x s.t. $x^n=a$.

Proof: (Existence)

Let $S:=\{s\in\mathbb{R}: s\geq 0, s^n<a\}$. Note that

① $S\neq\emptyset$ since $0\in S$.

② S is bounded above by $1+a$.

(Since if $s>1+a$, then $s^n>(1+a)^n>1+an>a$.)

By Supremum property, S has a supremum.

Let $x:=\sup S\geq 0$.

If we can show $x^n=a$, then of course $x>0$.

Hence we want rule out two bad cases

$x^n>a$ and $x^n<a$.

We will use this elementary inequality: if $0 \le a \le b$, then

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2} \cdot a + \cdots + b a^{n-2} + a^{n-1}) \le (b-a) \cdot n b^{n-1}.$$

Case 1: Suppose $x^n > a$.

Want: $(x - \frac{1}{m})^n > a$ for some large m.

Then $x - \frac{1}{m}$ is also an upper bound for S

Contradicting to the fact that $x = \sup S$.

Note that

$$x^n - (x - \tfrac{1}{m})^n \le \tfrac{1}{m} \cdot n x^{n-1}$$

By Archimedean Property, $\exists\, m \in \mathbb{N}$ s.t.

$$\frac{1}{m} < \frac{x^n - a}{n x^{n-1}} < x. \qquad (x^n > a)$$

Hence, we have

$$x - \tfrac{1}{m} > 0 \quad \text{and} \quad (x - \tfrac{1}{m})^n > a.$$

If $t > x - \frac{1}{m}$, then $t^n > (x - \frac{1}{m})^n > a^n$.

Hence $t \notin S$. Thus $x - \frac{1}{m}$ is an upper bound for S.

Case 2: Suppose $x^n < a$.

Want: $(x+\frac{1}{m})^n < a$ for some large m.

Then $x+\frac{1}{m} \in S$.

Contradicting to the fact that $x = \sup S$.

Note that

(*) $\quad (x+\frac{1}{m})^n - x^n \leq \frac{1}{m} \cdot n (x+\frac{1}{m})^{n-1} \leq \frac{n}{m}(x+1)^{n-1}$

By Archimedean Property,

$$\exists\, m \in \mathbb{N} \text{ s.t. } \frac{1}{m} < \frac{a-x^n}{n(x+1)^{n-1}} . \quad (x^n < a)$$

Hence, $\quad 0 \leq x < x+\frac{1}{m}$ and $(x+\frac{1}{m})^n < a$.